

INFINITE DIMENSIONAL INTEGRALS BEYOND MONTE CARLO METHODS: YET ANOTHER APPROACH TO NORMALIZED INFINITE DIMENSIONAL INTEGRALS

JEAN-PIERRE MAGNOT

ABSTRACT. An approach to (normalized) infinite dimensional integrals, including normalized oscillatory integrals, through a sequence of evaluations in the spirit of the Monte Carlo method for probability measures is proposed. In this approach the normalization through the partition function is included in the definition. For suitable sequences of evaluations, the ("classical") expectation values of cylinder functions are recovered

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1. INTRODUCTION

Let us first recall two well-developed frameworks.

- Let (X, μ) be a measured space. Following [9], [11], [12], let us fix a vector subspace $\mathcal{F} \subset L^\infty(X, \mu)$ such that $1_X \in \mathcal{F}$. A **mean** on \mathcal{F} is a linear map $\phi : \mathcal{F} \rightarrow \mathbb{C}$ such that $\phi(1_X) = 1$. Alternately, if (X, d) is a metric space, given $\mathcal{F} \subset C_b^0(X)$ (space of bounded maps), a **mean** on \mathcal{F} is a linear map $\phi : \mathcal{F} \rightarrow \mathbb{C}$ such that $\phi(1_X) = 1$. These two terminologies come from the basic example where μ is a Borel probability measure on a metric space (X, d) , for which the mean of a continuous integrable map f is its expectation value and can be approximated by sequences of barycenters of Dirac measures via Monte Carlo method, namely, via some sequences $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that

$$\forall f \in L^1(X, \mu), \int_X f d\mu = \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n f(x_k).$$

The technical condition for such a sequence is the following: for each μ -measurable set A , $\mu(A) = \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n 1_A(x_k)$.

- the Feynmann-Kac's formula: It is heuristic in the original Feynman's work, and a very difficult question is to give it a mathematical (rigorous) sense. Many approaches have been developed in which the following heuristic integral is central:

$$\frac{1}{\int e^{-iS} d\lambda} \int f e^{-iS} d\lambda$$

where S is the action functional of the physical theory; f is \mathbb{C} -valued prescribed map defined on an infinite dimensional vector space of configurations; λ is a heuristic infinite dimensional Lebesgue measure, that is a translation invariant measure on the space of configurations; $\int e^{-iS} d\lambda$ is a so-called "normalization constant", called "partition function" and sometimes noted as $Z(S)$, which can be understood as the total volume of the heuristic measure $e^{-iS} \lambda$, of "density" e^{-iS} with respect to λ ; and the whole formula stands as a mean value of f , called "expectation value" because taken with respect to the (heuristic) probability measure $\frac{1}{\int e^{-iS} d\lambda} e^{-iS} \lambda$.

This too short exposition on Feynman-Kac formula is not, of course completely satisfying, compared to the huge litterature on it and the wide variety of tentatives of rigorous definitions. We focus here on the theory of Fresnel integrals, which are rigorously defined. The reader can refer to [1],[2], [3], [4], [5], [7], [8], [13].

The technical features of the finite measure setting are almost the same as the ones of compact spaces. This suggests that, getting into the setting of infinite dimensional integrals, such as heuristic integrals of the Feynman-Kac formula, where straight way computations of the normalization constant (the partition function integral) lead to divergent approximations that need to be renormalized (see e.g. [2]), one can expect the same kind of problems in generalizing e.g. a Daniell integral to more complex theories as the problems that occur while passing from compact operators to bounded (or even unbounded) operators acting on a Hilbert space. With [10], we began a research program where the normalized integrals with respect to infinite volume measures are seen as particular means (which are means spanned by finite measures), and are not measures in the strictly speaking sense. This approach is coherent with the standard definition of Fresnel integrals (see e.g. [2]) which are defined through sequences of complex measures and normalizing weights. The goal of this communication is to define a similar approach with complex measures spanned by Dirac measures, and to compare with the classical approach of integration via cylinder functions.

2. SEQUENCES ON A SEPARABLE SPACE AND DIRAC MEANS

Let X be a separable topological separable space. Given $x \in X$, the evaluation map $\delta_x : f \in \mathbb{C}^X \mapsto f(x)$ is viewed as a Dirac measure in a probabilistic way. These measures are the extremals of the convex set of Borel probability measures $P(X)$. Let us define the affine space spanned by these elements.

Definition 2.1. A complex (resp. real) **Dirac mean** is a linear map $\tau : \mathcal{D}_\tau \subset C_b^0(X) \rightarrow \mathbb{C}$ which is defined as the limit of barycenters with complex (resp. real) weights of a sequence of Dirac measures on X , i.e. for $\mathbb{K} = \mathbb{R}$ or \mathbb{C} ,

$$\exists (x_n, \alpha_n)_{n \in \mathbb{N}} \in (X \times \mathbb{K})^{\mathbb{N}}, \forall m \in \mathbb{N}^*,$$

$$\left\{ \sum_{n=0}^m \alpha_n \neq 0 \right\} \quad \wedge \quad \left\{ \forall f \in C_b^0(X), \tau(f) = \lim_{m \rightarrow +\infty} \frac{1}{\sum_{n=0}^m \alpha_n} \left(\sum_{n=0}^m \alpha_n \delta_{x_n}(f) \right) \right\}.$$

We note by $\widetilde{\mathcal{DM}}_{\mathbb{K}}(X)$ the space of \mathbb{K} -Dirac means, by $\mathcal{DM}_{\mathbb{K}}(X)$ the set of Dirac means τ such that $\mathcal{D}_\tau = C_b^0(X)$, by $\widetilde{\mathcal{DM}}_{\mathbb{R}}^+(X)$ the means τ obtained by a sequence $(\alpha_n)_{n \in \mathbb{N}} \in \mathbb{R}_+^*$ and by $\mathcal{DM}_{\mathbb{R}}^+(X)$ the space $\widetilde{\mathcal{DM}}_{\mathbb{R}}(X) \cap \widetilde{\mathcal{DM}}_{\mathbb{R}}^+(X)$.

Examples.

- *Linear extensions of the limit* : Let $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} x_n = x$. Let $(\alpha_n)_{n \in \mathbb{N}} \in (\mathbb{R}_+^*)^{\mathbb{N}}$ such that $\liminf_{n \rightarrow +\infty} \alpha_n > 0$. Then, applying the results of [10], we get that the corresponding Dirac mean τ equals to the limit at x . By its definition, τ is defined on a wider class of functions, and then defines a linear extension of the limit.
- *Monte Carlo method extended* : Let $\mu \in P(X)$ and let $(x_n)_{n \in \mathbb{N}}$ such that $\forall f \in L^1(X, \mu)$, $\int_X f d\mu = \lim_{n \rightarrow +\infty} \frac{1}{n+1} \sum_{k=0}^n f(x_k)$. Then for any finite measure $\mu' = \varphi \cdot \mu$,

where $\varphi \in L^\infty$ is the μ -density of μ' , we have

$$\frac{1}{\mu'(X)} \int_X f d\mu' = \lim_{n \rightarrow +\infty} \frac{1}{\sum_{k=0}^n \varphi(x_k)} \sum_{k=0}^n \varphi(x_k) f(x_k).$$

If μ' is only σ -finite, the density φ is no longer bounded but the same formula defines $\tau \in \mathcal{DM}_{\mathbb{R}}(X)$ by

$$\tau(f) = \lim_{n \rightarrow +\infty} \frac{1}{\sum_{k=0}^n \varphi(x_k)} \sum_{k=0}^n \varphi(x_k) f(x_k).$$

For example, if $X = [0; 1]$ equipped with the Lebesgue measure λ and $\Psi : X \rightarrow \mathbb{R}$ be a smooth diffeomorphism such that $\Psi' > 1$. Let $\mu' = \Psi^*(\lambda)$. Then, the density φ of μ' with respect to λ on $[0; 1]$ is bounded below by 1, and hence the corresponding mean τ is a linear extension of the limit at infinity [10].

- *Means on a Hilbert space and on its unit sphere:* A Hilbert space has dense sequences $(x_n)_{n \in \mathbb{N}}$ which are mimicking, by their topological properties, the density property of the sequences used in the classical Monte-Carlo method. Then with the same formula as before, one can take:

$$\tau(f) = \lim_{m \rightarrow +\infty} \frac{1}{\sum_{n=0}^m \alpha_n} \left(\sum_{n=0}^m \alpha_n \delta_{x_n}(f) \right)$$

for an adequate function f . We shall discuss a desired class of sequences $(\alpha_n)_{n \in \mathbb{N}}$ next section. But what we have to remark is the following: the unit sphere S of the Hilbert space has the same properties, and then carries also such means. One can even wonder whether radial projections are possible, and emphasize some invariance under the action of an orthogonal group. We also remark that many efforts have been made in the context of metric geometry (see e.g. [9], [12]) to study means on the sphere. We leave open the question of such a spherical integration, and its significance for integration on a ball or on the Hilbert space through a spherical-like procedure.

3. NORMALIZED INFINITE DIMENSIONAL INTEGRALS AS MAPPINGS ON SEQUENCES

Here we come to the heuristic integral $\frac{1}{\int e^{-iS} d\lambda} \int f e^{-iS} d\lambda$ on a Hilbert space H . Here, since the part e^{-iS} stands as a density, we can make the following definition:

Definition 3.1. We define a function

$$I : H^{\mathbb{N}} \rightarrow \mathcal{DM}_{\mathbb{C}}(H)$$

$$(x_n)_{n \in \mathbb{N}} \mapsto \tau = \lim_{m \rightarrow +\infty} \frac{1}{\sum_{n=0}^m e^{-iS(x_n)}} \left(\sum_{n=0}^m e^{-iS(x_n)} \delta_{x_n} \right)$$

Since we have complex coefficients $e^{-iS(x_n)}$, we have to take care about the normalization constant which can be zero for “bad” sequences. So that, the domain of I cannot be $H^{\mathbb{N}}$. Then comes the analysis, beyond the convergence of the Monte Carlo method for classical probability measures. Let us quote some open questions:

- Which kind of sequence lead to invariance by a group acting on H ?
- In this general setting, what will be the status of 1-Lipschitz functions that play a crucial role in the approach by metric geometry?
- Which sequences lead to (true) measures on H ?

- For which perturbations of the action fonctionnal do we have some adequate Taylor expansions?

The list of open questions can be very long, but we wish to finish by an answer and an open perspective.

4. LINK WITH THE "CLASSICAL" MONTE CARLO METHOD AND PERSPECTIVES

Let us begin with integration of cylinder functions on the infinite cube $[0; 1]^{\mathbb{N}}$ (Daniell integral). (heuristic) Wick rotation in order to get a positive measure. Let us consider now a cylinder function f . Let P be a finite dimensional projection such that $f = f \circ P$ and let $S_P = S \circ P$. Then, adequate sequences for the Monte Carlo method are those whose push-forward on $[0; 1]^{\dim \text{Im} P}$ are also adequate for this method. Taking now a creasing sequence of orthogonal projectors P_k converging (weakly) to identity, the condition on the sequence $(x_n)_{n \in \mathbb{N}}$ is that for each $k \in \mathbb{N}$, the push-forwards of the sequences $(P_k(x_n))_{n \in \mathbb{N}}$ on $[0; 1]^{\dim \text{Im} P_k}$ fit with the desired conditions. A sequence $(x_n)_{n \in \mathbb{N}}$ for such a method exists, through e.g. the powers of π . Taking an action fonctionnal S and considering a finite product measure e^{-S} (after Wick rotation), the classical approach of the (classical) Monte-Carlo method is to pull-back the sequences on $[0; 1]^{\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$ through the coordinatewise pull-back of a product measure. This approach carries no additive weight $(\alpha_n)_{n \in \mathbb{N}}$. For Fresnel integrals, the density e^{-iS} is approximated by a density ξe^{-iS} where the unction ξ is chosen to get two convergent integrals $\int \xi e^{-iS} f d\lambda$ and $\int \xi e^{-iS} d\lambda$. Assuming ξ integrable by itself and defined as a product function, there is also (after normalization) a possible pull-back of a sequence $(x_n)_{n \in \mathbb{N}}$ adapted for the Monte Carlo method, and the weight we get is only $\alpha_n = e^{-iS(x_n)}$. We recover here an old problem, already quoted in [10]: the "Lebesgue" measures on $\mathbb{R}^{\mathbb{N}}$ have been extensively studied in the 40's but contain very few finite measure subsets (see e.g. [6] for an up-to-date exposition). As for the example described in [10], the theory of means is an attractive candidate to complete the theory of probabilities in such infinite dimensional problems. Unfortunately for applications, analysis on such objects has to be developed, and the topologies of the space \mathcal{DM} have to be studied.

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